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A Note on Effect of Sample Size on Operating Characteristic Curves of Process Capability Indices under Normality

Suboohi Safdar

Abstract— Process capability indices have been topics of the significance for statistically controlled processes under diverse distributional conditions since few decades. Process capability indices are the numerical quantities that reflect the functional relationships between allowable spread and the actual spread. For normal processes allowable spread is the function of preset limits and actual spread is the function of process parameter(s) which characterize the normal process sufficiently. This paper is a note on effect of sample size along with critical values on few aspects as proportion of non-con-conforming, process yields and operating characteristic curves of three basic generation indices Cp, Cpk and Cpm under normality. Process capability indices have always been contributing their significance in many areas of social and medical sciences where it is the task of quality practitioners and product designers to maintain the controlled processes and desired to know their capability in prospects.

Index Terms— Basis Capability Indices, sample size, proportion of non-conforming, process yields, Operating Characterstic curve, normal process.

1 Introduction

PROCESS capability indices are quantitiave and dimensioless measures that shows functional relationship between prest specification(s) and process parameter(s) of statistically controlled process (es). It has been the topic of interest from the last decade of researchers and quality practitioners. For normal processes see for details Kotz and Johnson [1], Juran [2], Kane [3], Chan et al. [4], Boyles [5], Pearn et al. [6] among many others. Deleryd [7] described process capability analysis in four steps. Maling Albing [8] further explained these steps as (i) Identify important characteristics Plan the study (ii) Establish statistical control, Gather data; (iii) Assess the capability of a process; (iv) Initiate improvement efforts. This paper is an inspiration of Deleryd steps which are essentials for quality practitioners and product designers working in various fields of social and medical sciences. For any existing and new processes it is always require that important features should identified to plan the study and make a process in statistical control to assess the capability of a process for continuous impovement. These substantial facets are proportion of non-conforming, process yields and sample size with critical values needed for maintaining the continuous improvements.

The paper is organized as follows. In section II the development and modification of three PCIs C_p, C_{pk} and C_{pm} with significant aspects as proportion of non-conforming, process yield, operating characterstic curves with relation of sample sizes and critical values along with downsides are discussed. In section III discussion and future recommendations are presented.

Dr. Suboohi Safdar is currently working as an Assistant Professor in Department of Statistics, Universiy of Karachi. Pakistan. E-mail: suboohi@uok.edu.pk

2 Process Capability Indices for normal populations: Development and modifications

Process capability indices illustrate the relation between the process parameter(s) and process specifications. For a statistically controlled process emanating from a normal population, process mean μ and process standard deviation σ are assumed to be enough sufficient parameters to describe a process as capable or incapable. Basic indices developed or modified for normal populations based on these two parameters μ and σ .

Now we discuss first generation index C_p with development and uses.

2.1 First generation index C_p

For a measured characteristic of a statistical controlled process exhibiting normal distribution as $N(\mu,\sigma^2)$ Juran [2] suggested first generation index named as C_p with one process parameter σ to determine process capability as

$$C_p = \frac{USL - LSL}{6\sigma}$$

Here LSL and USL are the lower and upper specification limits and therefore the difference USL-LSL is the maximum allowable spread. The quantity 6σ is sometime called the actual spread of process.

For $d = (USL - LSL)/2$, half-length of specification interval, index Cp may define as

$$C_p = \frac{d}{3\sigma} \quad (1)$$

The main purpose of process control is to make C_p large to authenticate the small values of actual spread; a similarly small amount of actual spread validates small process standard deviation which is always admired by quality practitioner.

2.1.1 Proportion of Non-Conforming

An aspect of C_p is proportion of non-conforming (NC) with relation of process yield in parts per million (PPM). As non-conforming item may a unit that is beyond the preset specification limits so the expected proportion 'p' of non-conforming product is its value outside the specification limits as; $p = P[X \notin [LSL, USL]]$

Since X is a normally distributed measurement so by standardizing we may have,

$$p = 1 - \left\{ \Phi\left(\frac{USL - \mu}{\sigma}\right) - \Phi\left(\frac{LSL - \mu}{\sigma}\right) \right\}$$

Here is cumulative distribution function of standard normal distribution.

Assuming estimated process mean as midpoint of specification limit we may get p as;

$$p = 1 - \left\{ \Phi\left(\frac{USL - \frac{LSL + USL}{2}}{\sigma}\right) - \Phi\left(\frac{LSL - \frac{LSL + USL}{2}}{\sigma}\right) \right\}$$

$$= 1 - \left\{ \Phi\left(\frac{USL - LSL}{2\sigma}\right) - \Phi\left(\frac{LSL - USL}{2\sigma}\right) \right\}$$

$$= 2\Phi\left[-\left(\frac{USL - LSL}{2\sigma}\right) \right]$$

$$p = 2\Phi\left(-\frac{d}{\sigma} \right)$$

From (1) expected proportion p of non-conforming product is therefore

$$p = 2\Phi(-3C_p) \quad (2)$$

Note that $2\Phi(-3C_p)$ is the minimum expected proportion of non-conforming items under normality so it can be used as PCI to know the status of a process as capable or incapable. If C_p =1 it is regarded 'acceptably small' because it does guarantee that under normality there will never be less than 0.27% of NC product.

Table 1 summarizes the quality conditions of selected valued of Cp, on the assumption of normal distribution with values of the minimum possible expected proportion 'p' of NC items in percentage and process yield in PPM.

TABLE 1
PROCESS YIELDS OF CP AGAINST VARIOUS QUALITY CONDITIONS

Quality Conditions	Cp	Proportion of NC	Yields PPM
Worst	0.33	31.73	317440
Inadequate	0.67	4.56	45600
Capable	1	0.27	2700
Satisfactory	1.33	0.01	63
Excellent	1.67	0.05	0.57
Super	2	0.06	0.001

Table 1 shows that as quality codition improves C_p increases and process yields decreases.

Table 2 lists the recommended threshold values of C_p for one-sided and two-sided specifications along with process yields in parts per million (PPM).

TABLE 2
RECOMMENDED VALUES OF CP FOR EXISTING & NEW PROCESS

Specifications	One-Sided		Two sided	
Process	**Cp**	**PPM**	**Cp**	**PPM**
Existing Process	1.25	88.42	1.3	66.07
New process	1.45	6.81	1.5	6.8
Critical Existing Process	1.45	6.81	1.5	6.8
Critical New Process	1.6	0.79	1.7	0.54

Source: Montgomery. Introduction to Statistical Quality Control, © 1985, p. 279. John Wiley & Sons, New York

2.1.2 OC-curve of C_p

Power of the test is the ability to reject the null hypothesis. It is the function of Type-II error. For C_p index $Power(C_p) = 1 - \beta(C_p)$, where $\beta(C_p)$ is the probability of accepting an incapable process. Burr [9] derived OC curve for testing standard deviation using the fact $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ Kane [3] proposed did for index C_p as an analogy of Burr derivation and provide tables for various combinations of sample sizes and c for testing the index C_p.

The following procedure is adopted for constructing OC-curve.

$H_0 : C_p \leq c$ Process fails to meet the capability requirement

$H_1 : C_p > c$ Process meets the capability requirement

Where c is the minimal precision value preset by product designer or may determine with following mathematical expression. We have α -risk as $\alpha = P\left\{\hat{C}_p > c \mid C_p = c\right\}$

$$\alpha = P\left\{\chi^2_{\alpha,n-1} < \frac{(n-1)C_p^2}{c^2}\right\}$$

The power of the index can be computed as

$$\pi(C_p) = P\left(\chi^2_{\alpha,\, n-1} < \frac{(n-1)C_p^2}{c^2}\right) \quad (3)$$

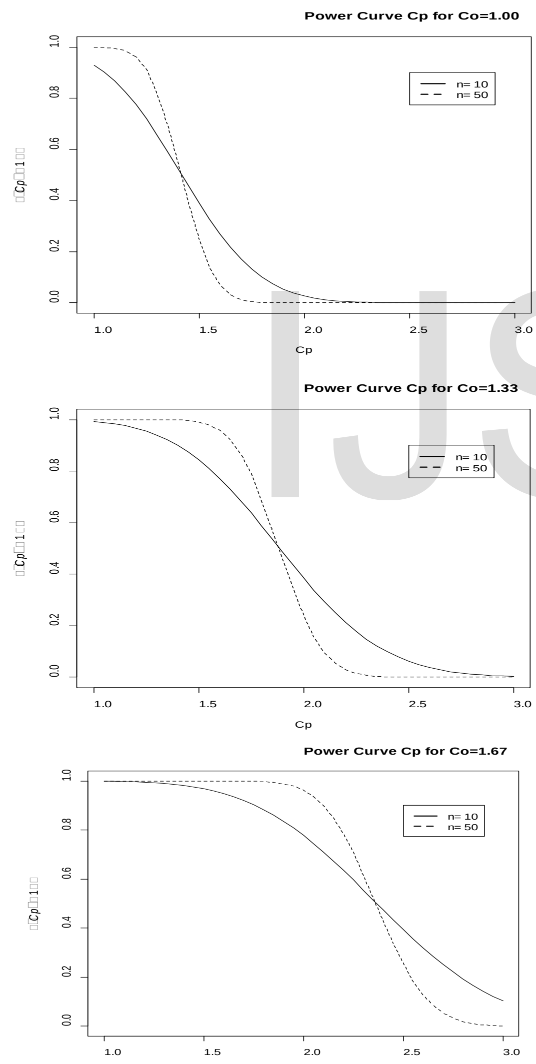

Figure 1: OC Curve for C_p for CO=1.00, 1.33, 1.67 and size of the sample

Figure 1 explain a plot of OC curve of an index C_p while values of C_p 1.0 (0.5) 3.0 are taken on x-axis and $\beta = 1 - \pi(C_p)$ on y-axis versus with three assumed critical values of C_0=1.00, 1.33, 1.67 for two size of the sample n=10 and 50.

We have noted the following from the plot of OC curve for index C_p

a. Size of the sample

With increasing size of the sample power increases and probability of committing Type II error decreases. So probability of accepting an incapable process decreases with increases sample size.

b. Critical value of C_O

Larger the critical value of c increasing the precision of power of test hence chance of accepting a process that do not meet the capability requirements decreases.

We have noted that C_p index alone cannot define process performance related with the process specifications as;

i. The design of C_p consist of only one potential performance parameter σ and irrespective of sample size, small values of C_p is a bad sign but large values of C_p do not guarantee of acceptability in the absence of information about process mean.

ii. This index is insensitive of process mean departure and only reflects process precision based on process variability. Hence for a unilateral specification with different means and constant variances C_p would result same values.

iii. The design of C_p makes comparison of the process spread to the engineering requirements only for two-sided specifications (bilateral) and is not useful for processes with one-sided specification (unilateral).

These downsides required some other indices that should consider process variability and also stand valid for unilateral specifications. We will now discuss development and modifications of second generation index C_{pk} to obviate these problems.

2.2 Second Generation Index C_{pk}

To preclude the weaknesses of C_p, especially ignorance of mean shift, Kane [3] introduced index C_{pk} which depends on both process mean μ and process standard deviation σ . C_{pk} is the ratio of the measured distance between the overall mean of the process and the closest specifications limit half of the total spread.

It shows how far the measurements of a process deviate from the mean

$$C_{pk} = \frac{\min\{USL - \mu, \mu - LSL\}}{3\sigma}$$

$$C_{pk} = \frac{1}{3\sigma}\left[d - \left|\mu - \frac{USL+LSL}{2}\right|\right] \quad (4)$$

2.2.1 Proportion of Non-Conforming

Boyles [5] showed that C_{pk} is a measure of process yield, but fails to distinguish between off target and on target process;

C_{pk} alone does not determine the proportion (p) of non-conformities as C_p does, but provides an upper bound given by $p \le 2\Phi(-C_{pk})$. Where $\Phi(.)$ denotes the cumulative probability function of standard normal distribution.
Boyles showed that the relationship between C_{pk} and process yield (1-p) is given by $2\Phi(3C_{pk})-1 \le 1-p < \Phi(3C_{pk})$. Where the relationship between C_{pk} and p is given $p_l = 1-\Phi(3C_{pk}) < p \le 2(1-\Phi(3C_{pk})) = p_u$. Hence expected proportion of NC product is $\Phi(-3C_{pk}) < E(p) < 2\Phi(-3C_{pk})$. However the exact expected proportion of NC product can be expressed in terms of two PCIs C_p and C_{pk} as follows

$$E(p) = \Phi(-3(2C_p - C_{pk})) + \Phi(-3C_{pk})$$

For capability index C_{pk} the probability of non-conformance is limited

$$p = 2\Phi(-3C_{pk}) \qquad (5)$$

For detail see Pearn, Kotz and Johnson [6].
Table 3 summarizes the quality conditions and proportion of NC in PPM with different C_{pk} values

TABLE 3
PROCESS YIELD OF CPK AGAINST QUALITY CONDITIONS

Quality Conditions	Cpk	Proportion of NC	Upper bound on NC in PPM
Worst	0.33	68.27	317311
Inadequate	0.67	95.46	45500
Capable	1	99.73	2700
Satisfactory	1.33	99.99	63
Excellent	1.67	99.9999	1
Super	2	99.9999998	0.002

Like C_p value of 1.0 for C_{pk} is acceptable but 1.33 is preferred because C_{pk}=1 pledge that proportion of conformities is not less than 99.73% or not greater than 0.27% see for detail Pearn and Chan [6]. To derive better estimates of p using C_p and C_{pk} Vannman and Albing [10] presented process capability plots see for more details Deleryd and Vannman [11] and Gabel [12]. Porter and Oakland [13] discussed the relationship between C_{pk} and the probability that a sample mean will fall outside control limits. Various tables are provided for different quality conditions versus accepted values of the index C_{pk}. Once the p is determined the proportion of non-conforming along with yield can calculated and hence different C_{pk} values can obtain the process yields.
Table 4 show the relationship between C_{pk} with expected number of non-conformities on both bounds.

TABLE 4
EXPECTED NUMBER OF NC ON LOWER AND UPPER BOUND OF CPK

C_{pk}	PL(PPM)	Pu (PPM)	Cpk	PL(PPM)	Pu (PPM)
0.1	382090	764180	1.1	480	970
0.2	274250	548500	1.2	160	320
0.3	184060	368120	1.3	50	100
0.4	115070	230140	1.4	10	30
0.5	66810	133620	1.5	0	10
0.6	35930	71860	1.6	0	0
0.7	17860	35720	1.7	0	0
0.8	8200	16400	1.8	0	0
0.9	3471	6942	1.9	0	0
1.0	1350	2700	2	0	0

2.2.2 OC CURVE OF C_{PK}

OC curve of C_{pk} explores the power of not accepting an incapable process. Pearn and Chen [14] proposed a hypothesis test to check that the process meets the capability requirement and runs under the desired quality condition

$$H_0: \; C_{pk} \le c$$

$$H_A: \; C_{pk} > c$$

α -risk is the chance of incorrectly accepting an incapable

$$\alpha = P\left[\tilde{C}_{pk} > c / C_{pk} = c\right]$$

$$\alpha = P\left[\frac{3n^{1/2}\tilde{C}_{pk}}{b_f} > \frac{3n^{1/2}c}{b_f}\right]$$

Now the α -risk can be determine as

$$\alpha = P\left[t_{n-1}(\delta) > \frac{3n^{1/2}c}{b_f}\right]$$

Here $t_{n-1,\alpha}(\delta_c) = 3\sqrt{n}c/b_f$ is the upper α quantile of $t_{n-1,\alpha}(\delta_c)$ distribution or $c = \frac{b_f}{3n^{1/2}} t_{n-1,\alpha}(\delta_c)$ with non-centrality parameter $\delta = 3n^{\frac{1}{2}}C_{pk}$

Pearn and Chan [14] provided the critical values for c=1.00, 1.33, 1.50, 2.00 n=10(40)250 and $\alpha = 0.01, 0.025, 0.05$ with values of correction factor b_f
The power of the test can be computed as

$$1-\beta = \pi(C_{pk}) = P\left[\tilde{C}_{pk} > C_o \mid C_{pk}\right]$$

$$\pi(C_{pk}) = P\left[t_{n-1}(\delta) > \frac{3n^{1/2}C_o}{b_f}\right] \qquad (6)$$

Figure 2 explain OC curves for index C_{pk} while C_{pk} 1.0 (0.1)

1.8 is taken on x-axis and $\beta = 1 - \pi(C_{pk})$ on y-axis. Critical value of c is set to be 1.00 and for knowing the power of the curve association we take α risk as 0.01 and 0.05.

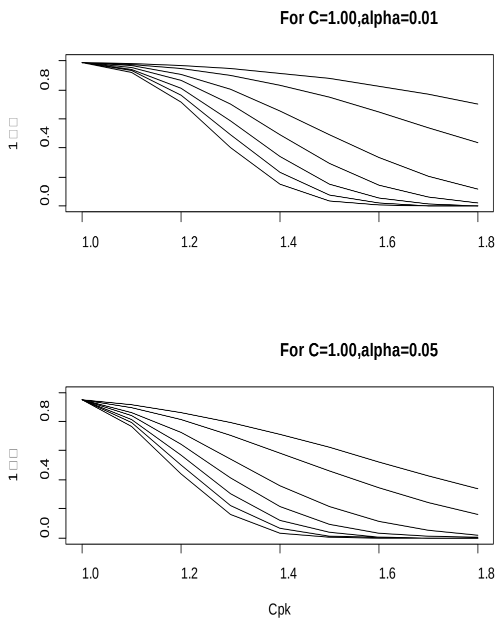

Figure 2: OC Curves for Cpk for n=10(30)250 Top to Bottom in Plot

We have noted the following from the plot of OC curve for index C_{pk}

a. SIZE OF THE SAMPLE

With increasing size of the sample probability of accepting an out of control process decreases.

b. $\alpha - Risk$

For small α power increases and larger the $\alpha - Risk$ smaller the chance of accepting an out of control process

According to quality improvement theories, it is important to use target values and to keep the process on target. (See, e.g. Bergman and Klefsjo [15].

i. The index C_{pk} takes the process mean into consideration but it may fail to distinguish between on-target processes from off target processes which is extremely useful in non-normal processes. In fact the design of C_p and C_{pk} are independent of target value T, which can fail to account for process targeting (the ability to cluster around the target)

ii. C_{pk} can only estimate the probability of creating a non-conforming component feature but cannot control the probability of creating a non-conforming product.

iii. For a set of normally distributed data with a non-zero mean, C_{pk} may over-estimate the fraction of non- conformance, especially when the value μ/σ is large enough.

iv. If the value(s) of measurements, comes from a stable normal population are outside the specification limits(s), C_{pk} in some cases could be negative and the process would then be inadequate for controlling the measurements of process(s).

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2.3 THIRD GENERATION INDEX C_{PM}

This index C_{pm} was first introduced Taguchi capability index in the published literature by Chan, Cheng, and Spiring [4]. It is measurable and directly related to the quadratic loss of the measured feature. Spiring et al. [4] also used the application of C_{pm} to a tool wear problem.
The index C_{pm} is defined as

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}} \qquad (7)$$

2.3.1 PROPORTION OF NON-CONFORMING

Ruczinski [16] lower bound on process yield is

$$\text{Yield} \geq 2\Phi(3C_{pm}) - 1 \qquad (8)$$

For $C_{pm} > \sqrt{3}/3$ an upper bound on the fraction of defectives can be defines as $\%NC \leq 2\Phi(-3C_{pm})$. Govaerts [17] found that $\text{Yield} \geq 1 - 2\Phi(-3C_{pm})$ holds for sufficiently large values of C_{pm}.

Table 5 summarizes the quality conditions of index C_{pm} with upper bound on non-conformities parts per million.

TABLE 5
PROCESS YIELD OF CPM AGAINST QUALITY CONDITIONS

Quality condition	C_{pm} values	NC(PPM)	Process Yield%
Inadequate	$C_{pm} < 1.00$	>2700	99.73
Capable	$1.00 \leq C_{pm} < 1.33$	66-2700	99.9934
Satisfactory	$1.33 \leq C_{pm} < 1.50$	8-66	99.9993
Excellent	$1.50 \leq C_{pm} < 2.00$	0.002-8	100

2.3.2 OC Curve for C_{pm}

When the given process meets the capability requirement and runs under the quality conditions, a hypothesis test may be applied to check that the process meets the capability requirement if $C_{pm} \le C$

$$H_0: \quad C_{pm} \le \tilde{C}_{pm}$$

$$H_A: \quad C_{pm} > \tilde{C}_{pm}$$

$$\pi(C_{pm}) = P\left[\tilde{C}_{pm} > C_o \mid C_{pm}\right]$$

$$OC(C_{pm}) = P\left[\chi'_{n,\frac{\alpha}{2}}\left(n\delta^2\right) < n\left(1+\delta^2\right)\left(\frac{C_{pm}}{\tilde{C}_{pm}}\right)^2\right] \qquad (9)$$

For convenience we used $\delta = 0, 1.0, 2, 2.5, 3, 3.5$ with sample sizes 10, 20, 30, 50,100 and plotted power curves for C_{pm} as shown in Figure 3

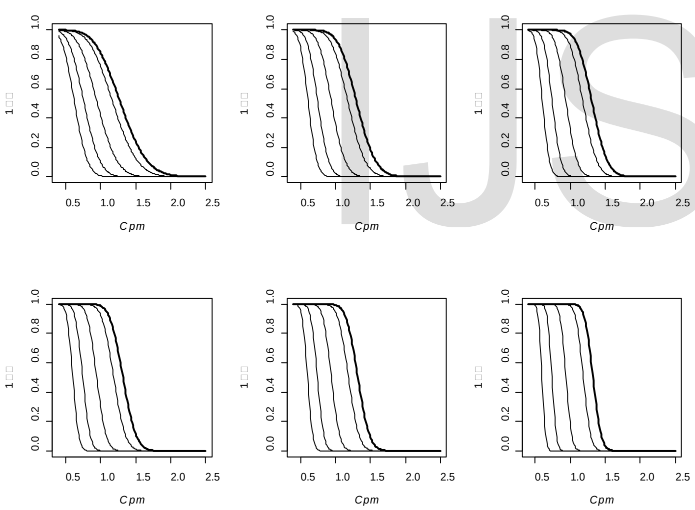

Figure 3: OC Curve of C_{pm} for n= 10,20,30,50,100 and $\delta = 0,1.0,2,2.5,3,3.5$

We have found the following drawbacks for the index C_{pm}

i. This index alerts the user when the process variance increases and the process mean deviates from its target value (or both) and estimate process capability around the target but cannot account an off-center process mean.

ii. The design of C_{pm} does not point out the closeness to either specification limit to the process mean as C_{pk} does but C_{pk} cannot take into account the off-center processes.

iii. It is observed that it is not necessary that a process that satisfy the capability requirement for index C_{pm} must endorse the capability requirement for index C_{pk} and a process that is found capable by C_{pk} must be proficient for Cpm which shows that C_{pm} do not provide the higher level of quality assurance with respect to process yield.

5 Discussion and Future Recommendation

For three basic capability indices under normality this paper is a comprehend addition to briefly understand the effect of various size of samples and critical values on proportion of nonconforming, process yields and operating characterstic curves. Almost in every discipiline of sciences where new and existing processes are performed it always have been the interest of product designers. It is the intention to work for these aspects of basic indices under non-normality.

References

[1] Kotz, S. and Johnson, N. L. (1993) Process Capability Indices, London: Chapman & Hall

[2] Juran, J.M. (1974) Juran's Quality Control Handbook. 3rd Edition. McGraw-Hill, New York.

[3] Kane, V.E. (1986) Process Capability Indices. Journal of Quality Technology, 18, 41-52.

[4] Chan, L.K., Cheng, S.W. and Spiring, F.A. (1988). A New Measure of Process Capability: C_{pm}. Journal of Quality Technology, 20, 162-175.

[5] Boyles, R. A. (1991). The Taguchi Capability Index. Journal of Quality Technology, 23, 107-126.

[6] Pearn, W.L., Kotz, S. and Johnson, N.L. (1992) Distributional and Inferential Properties of Process Capability Indices. Journal of Quality Technology, 24, 216 -231.

[7] Deleryd, M., (1998) On the Gap between Theory and Practice of Process Capability Studies. International Journal of Quality and Reliability Management, 15, 178-191.

[8] Albing M., (2008) Contribution to Process Capability Indices and Plots. PhD Thesis, Lulea University of Technology. Lulea Sweden

[9] Burr, I.R. (1976) Statistical Quality Control Methods. Marcel Dekker, New York, New York

[10] Vannman K. and Albing, M. (2007) Process Capability Plots for One-sided Specification Limits. Quality Technology and Quantitative Management, 4, 569-590.

[11] Deleryd, M. and Vannman, K. (1998) Process Capability Studies for Short Production Runs. International Journal of Reliability, Quality and Safety Engineering, 5(4), 383-401.

[12] Gabel, S. H. (1990) Process Performance Chart, ASQC Quality Congress Transactions -San Francisco, CA. 683-688.

[13] Porter, L. J. and Oakland, J. S. (1990) Measuring Process Capability using Indices - Some New Considerations. Quality and Reliability Engineering International, 6, 19-27.

[14] Pearn, W. L, Chen, K. S. (1999) Making Decisions in Assessing Process Capability Index Cpk. Quality and Reliability Engineering International, 15, 321-326

[15] Bergman, B. and Klefsjo, B (2003) Quality from Customer Needs to Customer Satisfaction (Lund, Sweden, Studentlitteratur)

[16] Ruczinski, I. (1996). The relation between Cpm and the degree of Includence Doctoral Dissertation, Univ. Wurzberg, Germany.

[17] Govaerts, B. (1994) Private Communications to S. Kotz.